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Unsteady forced convection heat transfer in a channel

JAMES SUCEC
Department of Mechanical Engineering, Rm 219 Boardman Hall, University of Maine, Orono, ME 04469, U.S.A.

and

DAVID RADLEY
Harris Graphics, Inc., Plattsburgh, NY 12901, U.S.A.

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Abstract—An exact analytical solution is found for unsteady heat transfer, in the first time domain, for a fluid flowing in a laminar, fully developed manner in a duct when the wall temperature is suddenly changed to $\theta_w = bx^n$. The solution to the governing partial differential equation is effected by the use of the Laplace transform and yields the transient surface heat flux as a function of time and position down the duct for $n = 1$–6. For comparison purposes, solutions are also obtained for a number of approximate models, namely, pure conduction, slug flow, quasi-steady, and a new model proposed by Sucec.

INTRODUCTION

UNSTEADY heat transfer to a fluid flowing in a duct is a problem which commonly arises in heat exchangers, jet engines, and nuclear reactors.

Perlmutter and Siegel [1], and Siegel [2], use the approximate integral method to solve the transient in a duct flow when the wall temperature is step changed or varies arbitrarily with time, respectively. Siegel and Perlmutter [3], by using a slug flow velocity profile, deal with unsteady channel flow with surface heat flux varying with time and position. Krishnan [4], by use of Laplace transforms, finds a solution in time domain I ($t < x/u_{max}$) to the conjugated problem of heat transfer in a pipe the outer wall of which is subject to a step change in either temperature or heat flux. Lin and Shih [5] use the approximate 'instant local similarity' method to analyze laminar flow with appreciable viscous dissipation in ducts when the wall temperature is abruptly changed. Their procedure leads to a solution valid at small times. Numerical finite difference methods are utilized by Chen *et al.* [6] and by Somasundaram *et al.* [7] to handle duct flow for step changes at the duct wall. Cotta and Ozisik [8] develop an approximate analytical solution with integral transforms for the case of a step change in pipe wall temperature when the flow inside the pipe is laminar.

In the present work, analytical solutions are considered for the problem of steady, laminar, constant property flow in a parallel plate duct when a transient is initiated by a sudden change in duct wall temperature to a power function in x, $\theta_w = bx^n$. The motivation for this work arose from an approximate analytical model for transient conjugated convection problems which is developed and tested in ref. [9]. As evidenced there, this model does very well for second time domain problems ($t > x/u_{max}$). However, succeeding work indicated some ambiguity in the model, along with poorer predictions in time domain I, $t < x/u_{max}$. So it was with the intent to develop a better approximate analytical model in this time domain to be used ultimately on conjugated problems that led to the exact, and approximate analytical solutions, in the present work. The wall temperature distribution used, $\theta_w = bx^n$, was selected to test the analytical model, developed for the first time domain, with a set of functions, 1, x, x^2, x^3, etc. which can be used to represent any other function by a series of such terms.

The approximate analytical solution developed here, as well as the predictions of other approximate models in the first time domain, will be compared with the exact analytical solution, found in the present work, for the transient surface heat flux and in some cases, the bulk mean temperature. These other approximate models are the slug flow model and the pure conduction model, both of which are discussed in Soliman and Johnson [10, 11], and the quasi-steady model.

Both the exact solution and the approximate analytical model developed can be applied to the general case of wall temperature being an arbitrary function of time and position along the duct.

ANALYSIS

The physical situation consists of a parallel plate duct with half height R through which a constant property fluid is flowing in a steady, laminar, fully developed fashion. Viscous dissipation effects and axial conduction within the fluid are negligible. The

NOMENCLATURE

a_0, a_1	coefficients in the velocity profile of equation (4)
b, B	coefficients in the wall temperature distribution as in equation (27)
C^j_m	coefficients defined in equation (18)
F	Fourier number, $\alpha t/R^2$
g_j	defined by equation (17)
h	surface coefficient of heat transfer
j	index
k	thermal conductivity of fluid
m	index
n	power on x in the wall temperature distribution
N	Nusselt number, hR/k
p	Laplace transform parameter
q_w	surface heat flux
Q	non-dimensional surface heat flux, $R\sqrt{\pi}\, q_w/kB$
R	half height of the parallel plate duct
t	time
T, T_i	local and initial temperature of the fluid, respectively
u, u_m, u_{max}	local, average, and maximum fluid velocity, respectively
x	space coordinate along duct
X	non-dimensional space coordinate, $\alpha x/R^2 u_m$
y	space coordinate perpendicular to duct wall
Y	non-dimensional space coordinate, y/R.

Greek symbols

α	thermal diffusivity of fluid
θ	local temperature excess, $T-T_i$
θ_B, θ_w	bulk mean and wall values of θ
σ	dummy integration variable for F
τ	$F-X$
ϕ_B	non-dimensional bulk mean temperature, $\sqrt{\pi}\,\theta_B/B$.

Superscript

$\bar{}$	Laplace transform with respect to F.

flowing fluid and duct walls are both at an initial constant temperature T_i when, suddenly, the duct wall temperature excess is changed to $\theta_w = bx^n$. The problem is to predict the time and space varying surface heat flux and the fluid's bulk mean temperature in the thermal entrance region of the duct during the first time domain.

With the temperature excess defined as $\theta(x, y, t) = T(x, y, t) - T_i$ and using the following non-dimensional independent variables, $F = \alpha t/R^2$, $X = \alpha x/R^2 u_m$, $Y = y/R$, the thermal energy equation for the problem is

$$\frac{\partial \theta}{\partial F} + \frac{u(Y)}{u_m}\frac{\partial \theta}{\partial X} = \frac{\partial^2 \theta}{\partial Y^2}. \tag{1}$$

The development of the solution to equation (1) will be carried out first for the general case of wall temperature being an arbitrary function of non-dimensional time, F, and axial coordinate X. In the first time domain, $F < 2X/3$, the fluid which was at the duct entrance at $F = 0$ has not yet reached the position X of interest and, hence, the inlet boundary condition is not relevant to the problem in this domain. Transforming to a new X-like independent variable, $\tau = F - X$, as per the development in ref. [9], gives the following mathematical problem statement:

$$\frac{\partial \theta}{\partial F} + \left[1 - \frac{u(Y)}{u_m}\right]\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial Y^2} \tag{2}$$

$$\theta(0, \tau, Y) = 0, \quad \theta(F, \tau, 0) = \theta_w(F, \tau)$$

$$\theta(F, \tau, Y \to \infty) = \text{finite}. \tag{3}$$

The velocity profile for laminar, fully developed flow in a parallel plate duct is given as

$$u(Y)/u_m = a_0 Y + a_1 Y^2 \tag{4}$$

where

$$a_0 = 3, \quad a_1 = -3/2.$$

It is noticed that even though the initial condition is independent of τ, and that the inlet boundary condition need not be satisfied in this first time domain, there is still dependence of the solution on the x-like coordinate τ because of the wall temperature dependency on τ. Thus, the problem is not a pure conduction problem, the convective transport term on the left-hand side of equation (2) must be retained and the velocity profile, equation (4), is needed in equation (2) and also in the evaluation of the bulk mean temperature.

To solve equation (2) subject to equation (3), the Laplace transform is taken with respect to dimensionless time F. Thus the temperature excess in the transform plane is defined as

$$\bar{\theta} = \int_0^\infty \theta\, e^{-pF}\, dF. \tag{5}$$

Taking the transform of equations (2) and (3) gives

$$\frac{\partial^2 \bar{\theta}}{\partial Y^2} - p\bar{\theta} = \left[1 - \frac{u(Y)}{u_m}\right]\frac{\partial \bar{\theta}}{\partial \tau} \tag{6}$$

$$\bar{\theta}(p, \tau, 0) = \bar{\theta}_w(p, \tau), \quad \bar{\theta}(p, \tau, Y \to \infty) = \text{finite}.$$

The initial approach to the solution of equation (6) was by successive approximations whereby an initial estimate of $\partial\bar{\theta}/\partial\tau$ is inserted into the right-hand side of equation (6), the resultant non-homogeneous partial differential equation is solved for $\bar{\theta}(p, \tau, Y)$ which is then inserted back into the right-hand side of equation (6) to give a second approximation to $\partial\bar{\theta}/\partial\tau$. This cycle is then continually repeated until two successive solution functions are close enough to each other. In this work, the initial approximation to $\partial\bar{\theta}/\partial\tau$ was taken as $\partial\bar{\theta}_w/\partial\tau$. Proceeding for four iterations it was noticed that a series solution of the following form was being developed:

$$\bar{\theta}(p,\tau,Y) = f_0(p,Y)\bar{\theta}_w + f_1(p,Y)\frac{\partial\bar{\theta}_w}{\partial\tau} + f_2(p,Y)\frac{\partial^2\bar{\theta}_w}{\partial\tau^2}\ldots \quad (7)$$

The coefficient functions, f_0, f_1, etc. had unchanging form with additional iterations as soon as any one of them had been used twice in the successive approximations procedure.

The heat flux in the transform plane is given by

$$\bar{q}_w = \frac{-k}{R}\left(\frac{\partial\bar{\theta}}{\partial Y}\right)_{Y=0}. \quad (8)$$

Equation (7) was inserted into equation (8) and the result was inverted back to the physical plane. This gave the following three integral term approximation to q_w:

$$q_w(F,\tau) = \frac{k}{R}\left\{\int_0^F \frac{1}{\sqrt{(\pi(F-\sigma))}}\frac{\partial\theta_w}{\partial\sigma}\,d\sigma - \int_0^F\left[\frac{a_1\sqrt{(F-\sigma)}}{2\sqrt{\pi}} + \frac{a_0}{4} - \frac{1}{2\sqrt{(\pi(F-\sigma))}}\right]\frac{\partial\theta_w}{\partial\tau}\,d\sigma - \int_0^F\left[\frac{\sqrt{(F-\sigma)}}{4\sqrt{\pi}} - \frac{a_0}{4}(F-\sigma) + \left(\frac{5a_0^2}{24} - \frac{a_1}{2}\right)\frac{(F-\sigma)^{3/2}}{\sqrt{\pi}} + \frac{9a_0a_1}{32}(F-\sigma)^2 + \frac{19a_1^2(F-\sigma)^{5/2}}{60\sqrt{\pi}}\right]\frac{\partial^2\theta_w}{\partial\tau^2}\,d\sigma + \cdots\right\}. \quad (9)$$

The first integral in equation (9) can be shown to be the exact analytical solution for a slug velocity profile, $u(Y) = u_m$. If the wall temperature distribution were independent of X, then $\partial\theta_w/\partial X = \partial\theta_w/\partial\tau = 0$ and all integrals vanish except for the first one. In this case, $\partial\theta_w/\partial X = 0$, equation (1) shows the problem to be a pure conduction problem which has the same exact solution as for the slug velocity profile, namely the first integral in the series given by equation (9). Thus the integrals beyond the first one in equation (9) give the effect of the actual non-slug velocity profile on the wall heat flux.

As an approximate analytical solution function for the transient heat flux, it is proposed to take the first two integrals giving

$$q_w = \frac{k}{R}\left\{\int_0^F \frac{1}{\sqrt{(\pi(F-\sigma))}}\frac{\partial\theta_w}{\partial\sigma}\,d\sigma - \int_0^F\left[\frac{a_1\sqrt{(F-\sigma)}}{2\sqrt{\pi}} + \frac{a_0}{4} - \frac{1}{2\sqrt{(\pi(F-\sigma))}}\right]\frac{\partial\theta_w}{\partial\tau}\,d\sigma\right\}. \quad (10)$$

The model of equation (10) eliminates the ambiguity in the analytical model in time domain I of ref. [9] and is proposed for use in both conjugated and non-conjugated problems. This model will be tested later in this work by comparing it to some exact solutions.

The four successive approximations which led to equation (7) suggest that the form of the solution in the transform plane becomes

$$\bar{\theta}(p,\tau,Y) = \sum_{j=0}^{\infty} f_j(p,Y)\frac{\partial^j\bar{\theta}_w}{\partial\tau^j} \quad (11)$$

where

$$\frac{\partial^0\bar{\theta}_w}{\partial\tau^0} \equiv \bar{\theta}_w.$$

Using equation (11) in differential equation (6), gives, after rearrangement

$$\left(\frac{d^2f_0}{dY^2} - pf_0\right) + \sum_{j=1}^{\infty}\left[\left(\frac{d^2f_j}{dY^2} - pf_j\right) - (1 - a_0Y - a_1Y^2)f_{j-1}\right]\frac{\partial^j\bar{\theta}_w}{\partial\tau^j} = 0. \quad (12)$$

Requiring that the coefficients of $\partial^j\bar{\theta}_w/\partial\tau^j$ vanish for all j yields the following set of ordinary differential equations for the f_j functions needed in the solution given by equation (11). Hence

$$\frac{d^2f_0}{dY^2} - pf_0 = 0 \quad (13)$$

$$\frac{d^2f_j}{dY^2} - pf_j = (1 - a_0Y - a_1Y^2)f_{j-1}. \quad (14)$$

The boundary conditions on equations (13) and (14) are given by

$$Y = 0, \quad \bar{\theta} = \bar{\theta}_w, \quad f_0 = 1, \quad f_j = 0, \quad j \geqslant 1$$
$$Y \to \infty, \quad f_j = \text{finite}. \quad (15)$$

With equations (13)–(15) in hand, it is no longer necessary to use successive approximations formally, instead these equations are used to solve directly for $f_0(p, Y)$, $f_1(p, Y)$, etc.

Solution of equation (13) and application of boundary conditions (15), yields

$$f_0 = e^{-\sqrt{p}Y}. \quad (16)$$

This is now inserted into equation (14) and f_1 is solved for and so on. Actually, much of the work needed to solve equation (14) for f_j when $j > 0$ can be reduced. A study of the structure of equation (14), the f_0 solution of equation (16), and the boundary conditions of equation (15) indicates that the functional form of every f_j is

$$f_j = g_j(Y)\,e^{-\sqrt{p}Y} \qquad (17)$$

where $g_j(Y)$ is a polynomial in Y of lowest order Y and highest order Y^{3j} the coefficients of which are dependent upon p. Thus, equation (17) can be further rewritten as

$$f_j(p, Y) = e^{-\sqrt{p}Y} \sum_{m=0}^{3j} C_m^j Y^m \qquad (18)$$

where C_m^j is the p dependent coefficient. Substitution of equation (18) into equation (14) gives the set of simultaneous algebraic equations

$$2m\sqrt{p}\,C_m^j - m(m+1)C_{m+1}^j = a_1 C_{m-3}^{j-1} + a_0 C_{m-2}^{j-1} - C_{m-1}^{j-1}, \quad m > 0. \qquad (19)$$

Also

$$C_0^0 = 1, \quad C_m^j \equiv 0 \quad \{m > 3j \text{ or } m < 1, j > 0\}. \qquad (20)$$

Combining equations (18) and (11) gives the solution in the transform plane as

$$\bar{\theta} = e^{-\sqrt{p}Y} \sum_{j=0}^{\infty} \left\{ \sum_{m=0}^{3j} C_m^j Y^m \right\} \frac{\partial^j \bar{\theta}_w}{\partial \tau^j}. \qquad (21)$$

Using equation (21) in equation (8) gives the heat flux in the transform plane

$$\bar{q}_w = \frac{-k}{R}\left\{ -\sqrt{p}\,\bar{\theta}_w + \sum_{j=1}^{\infty} C_1^j \frac{\partial^j \bar{\theta}_w}{\partial \tau^j} \right\}. \qquad (22)$$

Equations (21) and (22) are the exact analytical solutions in the transient thermal entrance region of time domain I, $F < 2X/3$, for an arbitrary duct wall temperature distribution $\theta_w = \theta_w(F, X)$, once the algebraic equations, equations (19), are solved for the C_m^j.

Equations (19) were solved by hand for $C_m^j(p)$ (the coefficients contain the transform parameter p in them) for $j = 1$–4 and then equation (22) was inverted to give the flux, q_w, in the physical plane as a sum of integral expressions, the first three of which were given earlier in equation (9). However, as j increases, the algebraic work needed to find q_w rapidly increases. To alleviate this problem, the analytical procedure was put on to the computer using a symbolic programming package called REDUCE. Symbolic programming allows symbolic mathematical operations, such as addition, multiplication, integration, and inversion, to be carried out yielding the same analytic, not numerical, expressions, as those done by hand. This was carried out to $j = 6$ giving a flux expression like that of equation (9), but with seven integrals instead of three. This expression for q_w, when $j = 6$, occupies almost two full pages of standard size typing paper and is available in ref. [12]. The test cases indicate that the approximate analytical expression of equation (10), which uses just two integrals is highly accurate and should suffice. The three integral expression has already been given as equation (9) and the fourth integral to be inserted inside the curly brackets of equation (9) is given as

$$-\int_0^F \left[\frac{-(F-\sigma)^{3/2}}{12\sqrt{\pi}} + \frac{a_0}{8}(F-\sigma)^2 + \left(\frac{a_1}{4} - \frac{5a_0^2}{24}\right)\frac{(F-\sigma)^{5/2}}{\sqrt{\pi}} + \left(\frac{5a_0^3}{128} - \frac{9a_0 a_1}{32}\right)(F-\sigma)^3 + \left(\frac{221 a_1 a_0^2}{840} - \frac{19a_1^2}{60}\right)\frac{(F-\sigma)^{7/2}}{\sqrt{\pi}} + \frac{51 a_1^2 a_0}{256}(F-\sigma)^4 + \frac{631 a_1^3 (F-\sigma)^{9/2}}{3780\sqrt{\pi}} \right] \frac{\partial^3 \theta_w}{\partial \tau^3}\,d\sigma. \qquad (23)$$

It must be kept in mind that the first integral appearing in equation (9) or (10) is a Stieltjes integral.

The bulk mean temperature of the fluid as a function of axial coordinate and time is found by use of its definition, namely

$$\theta_B = \int_0^1 \frac{u(Y)}{u_m}\,\theta(F, \tau, Y)\,dY. \qquad (24)$$

Taking the Laplace transform with respect to F of equation (24), and changing the upper limit because of the thermal entrance region being considered, yields

$$\bar{\theta}_B = \int_0^{\infty} (a_0 Y + a_1 Y^2)\bar{\theta}(p, \tau, Y)\,dY. \qquad (25)$$

Inserting the expression for $\bar{\theta}$, equation (21), into equation (25) gives the transformed bulk mean temperature. Taking the inverse Laplace transformation produces the bulk mean temperature in the physical plane when three terms of the series in equation (21) are used

$$\theta_B(F, \tau) = \int_0^F \left[a_0 + \frac{4a_1\sqrt{(F-\sigma)}}{\sqrt{\pi}} \right] \theta_w(\sigma, \tau)\,d\sigma + \int_0^F \left\{ -a_0(F-\sigma) + \left(\frac{8}{3}a_0^2 - 4a_1\right)\frac{(F-\sigma)^{3/2}}{\sqrt{\pi}} + \frac{27}{4}a_0 a_1 (F-\sigma)^2 + \frac{44a_1^2(F-\sigma)^{5/2}}{3\sqrt{\pi}} \right\} \left(\frac{\partial \theta_w}{\partial \tau}\right)_\sigma d\sigma + \int_0^F \left\{ \frac{a_0(F-\sigma)^2}{2} + \left(2a_1 - \frac{8}{3}a_0^2\right)\frac{(F-\sigma)^{5/2}}{\sqrt{\pi}} + \left(\frac{5}{4}a_0^3 - \frac{27}{4}a_0 a_1\right)(F-\sigma)^3 + \left(\frac{1651 a_0^2 a_1}{105} - \frac{308a_1^2}{21}\right)\frac{(F-\sigma)^{7/2}}{\sqrt{\pi}}\right.$$

$$+\frac{1461}{64}a_0a_1^2(F-\sigma)^4$$

$$+\frac{11\,774a_1^3(F-\sigma)^{9/2}}{315\sqrt{\pi}}\Bigg\}\left(\frac{\partial^2\theta_w}{\partial\tau^2}\right)_\sigma d\sigma+\cdots \quad (26)$$

Solutions for $\theta_w = bx^n$

As is evident by inspection of equations (21) and (22), exact solutions in closed form are possible when

$$\frac{\partial^j\bar{\theta}_w}{\partial\tau^j}=0 \quad \text{for } j\geqslant \text{a finite integer such as } n.$$

A class of wall temperature distributions which satisfy this condition is $\theta_w = bx^n$. This class, as mentioned earlier, also allows the study of the probable performance of our approximate analytical two-term model, equation (10), on more general surface temperature distributions which are, in effect, composed of these fundamental harmonics, 1, x, x^2, $x^3,\ldots$, $x^n,\ldots$ which can serve as base vectors in function space in the same role performed by sine functions in a Fourier sine series expansion of an arbitrary function.

The wall temperature distribution in the F, τ variables required in the analytical expressions for q_w is given as

$$\theta_w = B(F-\tau)^n. \quad (27)$$

Inserting the needed derivatives of θ_w into the version of equation (9) which contains seven integrals, for $n = 1$–6 in equation (27), yields the exact solution for the wall flux since $\partial^j\bar{\theta}_w/\partial\tau^j = 0$ for $j > n$, for these six wall temperature distributions. Only the expressions for the non-dimensional flux Q for $n = 1$ and 2 will be displayed here while the results in the figures will be for all six cases:

$n = 1$

$$Q=[-\tau F^{-1/2}+2F^{1/2}]+\left[\frac{a_1}{3}F^{3/2}-F^{1/2}+\frac{a_0\sqrt{\pi}F}{4}\right]; \quad (28)$$

$n = 2$

$$Q=\left[\tau^2F^{-1/2}+\frac{8}{3}F^{3/2}-4\tau F^{1/2}\right]$$

$$+\left[\frac{4a_1}{15}F^{5/2}-\frac{2a_1}{3}\tau F^{3/2}-\frac{4}{3}F^{3/2}+2\tau F^{1/2}\right.$$

$$\left.+\frac{a_0\sqrt{\pi}}{4}F^2-\frac{a_0\sqrt{\pi}}{2}\tau F\right]+\left[-\frac{19a_1^2}{105}F^{7/2}\right.$$

$$+\left(\frac{2a_1}{5}-\frac{a_0^2}{6}\right)F^{5/2}-\frac{F^{3/2}}{3}$$

$$\left.-\frac{3a_1a_0\sqrt{\pi}}{16}F^3+\frac{a_0\sqrt{\pi}}{4}F^2\right]. \quad (29)$$

In equations (28) and (29), the square brackets separate the contributions of the individual integrals of the seven-term version of equation (9). Thus, the exact solution contains only two terms for $n = 1$ since $\partial^j\bar{\theta}_w/\partial\tau^j = 0$ for $j > 1$ and, in equation (29) the three terms shown are the exact solution since derivatives with respect to τ under the integral signs vanish for $j > 2$.

Next, the bulk mean temperature is evaluated for $n = 1$ and 2 in equation (27) by use of equation (26) which yields the following expression for $n = 1$. The result for $n = 2$ is not shown for the sake of brevity.

For $n = 1$, $\theta_w = bx = B(F-\tau)$

$$\theta_B = B\left[\frac{a_0F^2}{2}-a_0\tau F+\frac{16a_1F^{5/2}}{15\sqrt{\pi}}-\frac{8a_1\tau F^{3/2}}{3\sqrt{\pi}}\right]$$

$$-B\left[\frac{88a_1^2F^{7/2}}{21\sqrt{\pi}}+\frac{9a_1a_0F^3}{4}\right.$$

$$\left.+\left(\frac{16a_0^2}{15}-\frac{8a_1}{5\sqrt{\pi}}\right)F^{5/2}-\frac{a_0F^2}{2}\right]. \quad (30)$$

In equation (30), the square brackets enclose the contributions of the individual integrals of equation (26).

The approximate solutions

As discussed earlier, the approximate analytical model being proposed in this work is the truncation of our exact solution, equation (9), after the first two integrals and is given as equation (10) for the heat flux and as the first two integrals of equation (26) for the bulk mean temperature.

The slug flow approximate model can be shown to be the first term of equation (9) for the heat flux.

The pure conduction approximate solution is found by setting $a_0 = a_1 = 0$, to give zero velocity in the duct.

The final approximate model to be tested is the quasi-steady solution employing the constant surface coefficient of heat transfer, h, appropriate to steady flow through a duct the wall of which is isothermal. Solution of an energy balance on the fluid in $\tau-F$ variables for $\theta_w = BX^n$ gives the quasi-steady bulk mean temperature as

$$\frac{\theta_B}{B}=\left[-(-\tau)^n+\frac{n}{N}(-\tau)^{n-1}-\frac{n(n-1)}{N^2}(-\tau)^{n-2}\right.$$

$$\left.+\frac{n(n-1)(n-2)}{N^3}(-\tau)^{n-3}+\cdots\right]e^{-NF}$$

$$+\left[(F-\tau)^n-\frac{n}{N}(F-\tau)^{n-1}+\frac{n(n-1)}{N^2}(F-\tau)^{n-2}\right.$$

$$\left.-\frac{n(n-1)(n-2)}{N^3}(F-\tau)^{n-3}+\cdots\right]. \quad (31)$$

The non-dimensional quasi-steady flux, Q, is given by

$$Q=\sqrt{\pi}\,N\left[X^n-\frac{\theta_B}{B}\right]. \quad (32)$$

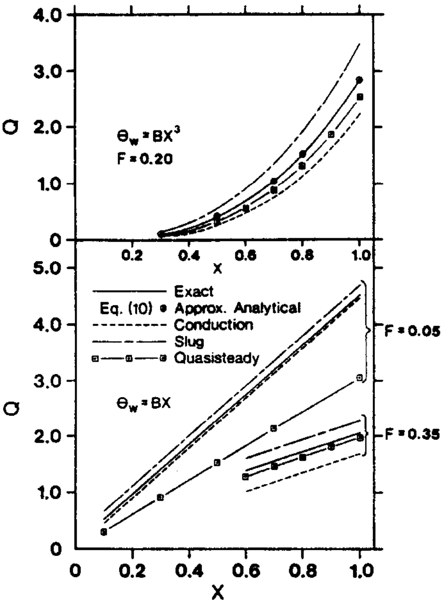

FIG. 1. Axial variation of flux at different times.

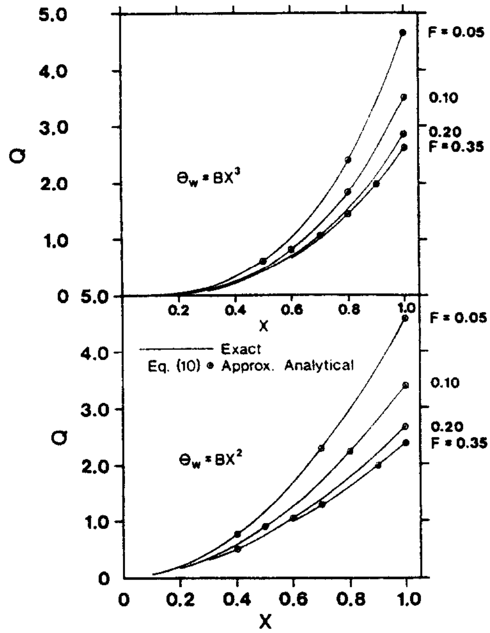

FIG. 2. Axial variation of flux at different times.

RESULTS AND DISCUSSION

By setting the coefficients in the velocity profile, equation (4), equal to their respective values for this problem, namely, $a_0 = 3$, $a_1 = -3/2$, in equations (28) and (29), and in equation (30), one obtains the exact solution for the flux and bulk mean temperature, respectively. For the surface heat flux, this was also carried out for $n = 3$–6 in equations which, because of their length, were not included here for the sake of brevity. Results are also obtained, for these same cases, from the approximate analytical model being proposed herein, namely, equation (10) for the flux and the first two integrals in equation (26) for the bulk mean temperature. Note that the exact solution functions, which can be applied to any arbitrary $\theta_w(F, X)$, and not just to the class $\theta_w = bx^n$ being considered here, are given by equations (9) and (26).

In addition, results were also found for the three approximate models used previously in the literature, namely, the slug flow, pure conduction and quasi-steady models.

Looking at the plotted results, at $F = 0.05$, for the flux when $\theta_w = bx$ in the lower half of Fig. 1, one sees trends for the various earlier approximate models which are also reflected in the figures for the other cases for n, $n = 2$–6. The slug flow model, because of its use of a velocity profile which gives velocities near the wall which are higher than the actual velocity profile, equation (4), yields a higher surface heat flux than does the exact solution and all the approximate models. The pure conduction model gives a smaller surface heat flux than the actual flux because of its use of zero velocity throughout the fluid thus removing the convective energy transport mechanism from the situation. The quasi-steady approach, the bottom curve in the figure at $F = 0.05$ gives a much smaller flux than actually occurs because of its use of the relatively low surface coefficient of heat transfer for steady flow through an isothermal duct. This error is particularly large at the low times, such as $F = 0.05$, because of the very high energy transfer rates due to a thin thermal boundary layer at low values of F. For this case, $\theta_w = bx$, the approximate analytical model of this work gives the exact result, and hence, plots right on top of the exact solution. This results from the fact that our two-term approximation, equation (10), is exact for the case of $n = 1$ and also, incidentally, is exact for the case of $n = 0$. At a large value of time, such as $F = 0.35$ in Fig. 1, the trends for the various approximate models are basically the same as previously described except for the quasi-steady solution which provides a better prediction than does the pure conduction solution.

For the cases of $n = 2$, 3, and 6 in equation (27), surface heat flux results for the exact solution, and for the approximate model of this work, are given in Figs. 1–4 for a number of different times F. Selected comparisons of the exact solution with the other approximate models are given in Figs. 1 and 4. As is evident from the figures, the approximate analytical model, even though it is not the exact solution for the cases of $n = 2$–6 as it was for $n = 1$, is essentially indistinguishable from the exact solution for the scale used on the graphs. Thus, only a few typical predictions, the circles, are shown for the approximate analytical solution. One can discern a slight difference between exact and approximate analytical solutions in Fig. 4 at $F = 0.35$ where the approximate analytical prediction is slightly higher than the exact results. Hence, the two-term representation of the flux in the approximate analytical model, equation (10), gives

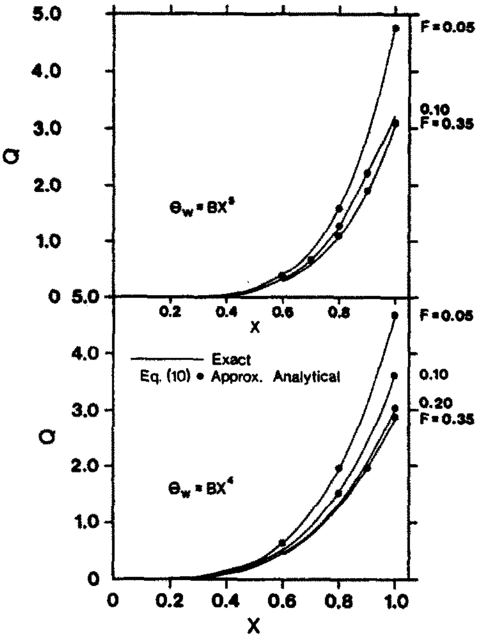

FIG. 3. Axial variation of flux at different times.

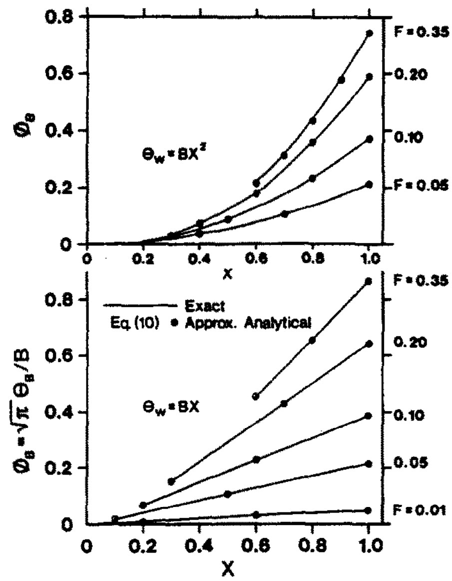

FIG. 5. Axial variation of bulk mean temperature.

excellent results even for the higher level harmonic of $n = 6$.

Looking at the predictions of the other approximate models for $n > 1$ in the upper half of Figs. 1 and 4, the general trends are the same qualitatively, as previously discussed for $n = 1$, but the errors are greater in magnitude for the higher values of n, such as $n = 6$, than for the case of $n = 1$.

The comparisons of the exact solution for the bulk mean temperature with the approximate analytical model of the present work is shown in Figs. 5 and 6. Again, the approximate model is also the exact solution when $n = 1$ and, as was also the case for the flux, is essentially coincident with the exact solution at the higher value of n. The other approximate models give significant error.

CONCLUSION

An exact analytical solution has been found for the problem being considered, namely, unsteady thermal entry heat transfer within a parallel plate duct in the first time domain with $\theta_w = bx^n$. An approximate ana-

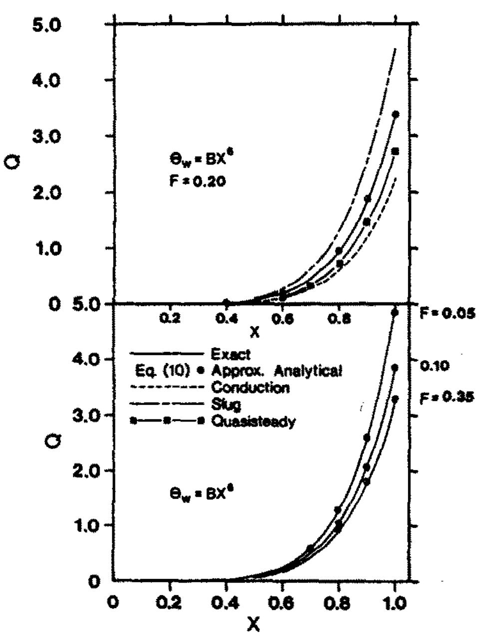

FIG. 4. Axial variation of flux at different times.

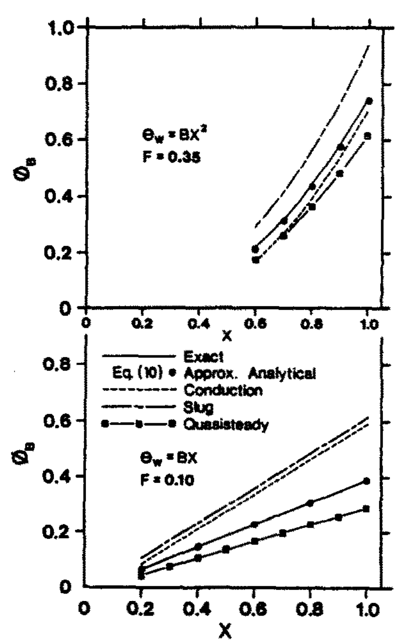

FIG. 6. Axial variation of bulk mean temperature.

lytical model has also been developed which yields results of very high accuracy, even for the higher level harmonic represented by $n = 6$ in $\theta_w = bx^n$. Hence, this model should be able to be employed with high accuracy on problems of arbitrary variation of wall temperature with F and X and, therefore, also on transient conjugated problems.

If there is interest in even higher accuracy or in checking the magnitude of the contributions of integral terms, beyond the first two, which are neglected in equation (10), equation (9) can be used. Equation (9) gives the exact solution for $\theta_w = bx^n$ using a finite number of terms and represents the exact solution for any $\theta_w(F, X)$ if one retains all the integrals in the infinite series.

It is found that the more traditional approximate models, slug, pure conduction, and quasi-steady, exhibit appreciable error in their predictions of flux and bulk mean temperature. An exception to this occurs at low times for the pure conduction model.

REFERENCES

1. A. Perlmutter and R. Siegel, Two dimensional unsteady incompressible laminar duct flow with a step change in wall temperature, *Int. J. Heat Mass Transfer* **3**, 94–107 (1961).
2. R. Siegel, Heat transfer in laminar flow in ducts with arbitrary time variations in wall temperature, *J. Appl. Mech. Trans. ASME* **27**, 241–249 (1960).
3. R. Siegel and M. Perlmutter, Laminar heat transfer in a channel with unsteady flow and wall heating varying with position and time, *J. Heat Transfer* **85**, 358–365 (1963).
4. B. Krishnan, On conjugated heat transfer in fully developed flow, *Int. J. Heat Mass Transfer* **25**, 288–291 (1982).
5. Hsiao-Tsung Lin and Yen-Ping Shih, Unsteady thermal entrance heat transfer of power law fluids in pipes and plate slits, *Int. J. Heat Mass Transfer* **24**, 1531–1539 (1981).
6. S. C. Chen, N. K. Anand and D. R. Tree, Analysis of transient laminar convective heat transfer inside a circular tube, *J. Heat Transfer* **105**, 922–924 (1983).
7. S. Somasundaram, N. K. Anand and S. R. Husain, Calculation of transient turbulent heat transfer in a rectangular channel: two-layer model, *Numer. Heat Transfer* **13**, 467–480 (1988).
8. R. M. Cotta and M. N. Ozisik, Transient forced convection in laminar channel flow with timewise variations of wall temperature, ASME Paper No. 85-WA/HT-72, Winter Annual Meeting, Miami Beach, 17–21 November (1985).
9. J. Sucec, An improved quasi-steady approach for transient, conjugated forced convection problems, *Int. J. Heat Mass Transfer* **24**, 1711–1722 (1981).
10. M. Soliman and H. A. Johnson, Transient heat transfer for turbulent flow over a flat plate of appreciable thermal capacity and containing time dependent heat source, *J. Heat Transfer* **89**, 362–370 (1967).
11. M. Soliman and H. A. Johnson, Transient heat transfer for forced convection flow over a flat plate of appreciable thermal capacity and containing an exponential time dependent heat source, *Int. J. Heat Mass Transfer* **11**, 27–38 (1968).
12. D. Radley, Unsteady heat transfer in a channel, M.S. Thesis, University of Maine, Orono, Maine (1986).

CONVECTION THERMIQUE FORCEE VARIABLE DANS UN CANAL

Résumé—On trouve une solution analytique exacte pour le transfert thermique variable, dans les premiers instants, pour un fluide s'écoulant de façon laminaire, pleinement établie dans une conduite, avec une température pariétale qui est brusquement changée selon $\theta_w = bx^n$. La solution de l'équation fondamentale aux dérivées partielles est obtenue par utilisation de la transformée de Laplace et elle fournit le flux thermique variable à la paroi en fonction du temps et de la position en aval pour $n = 1$ à 6. Pour permettre des comparaisons, des solutions sont aussi obtenues pour plusieurs modèles approchés et nomément, conduction pure, écoulement piston, état permanent et un nouveau modèle proposé par Sucec.

INSTATIONRÄRE ERZWUNGENE KONVEKTION IN EINEM KANAL

Zusammenfassung—Der instationäre Wärmeübergang in einer laminaren, voll ausgebildeten Strömung in einem Kanal, dessen Wandtemperatur sich plötzlich gemäß $\theta_w = bx^n$ ändert, wird für einen ersten Zeitabschnitt exakt analytisch berechnet. Die zugrundeliegende partielle Differentialgleichung wird durch die Anwendung der Laplace-Transformation gelöst. Dabei ergibt sich die zeitlich veränderliche Oberflächenwärmestromdichte als Funktion von Zeit und Ort entlang des Kanals für $n = 1$ bis 6. Zu Vergleichszwecken werden auch Lösungen für eine Anzahl von Näherungsmodellen bestimmt, nämlich für reine Wärmeleitung, Kolbenströmung, quasi-stationären Zustand sowie für ein neues Modell, das von Sucec vorgeschlagen worden ist.

ТПЛОПЕРЕНОС ПРИ НЕСТАЦИОНАРНОЙ ВЫНУЖДЕННОЙ КОНВЕКЦИИ В КАНАЛЕ

Аннотация—Получено точное аналитическое решение для начальной стадии нестационарного теплопереноса жидкости в условиях ламинарного полностью развитого течения в канале при мгновенном изменении температуры стенки до $\theta_w = bx^n$. Основное дифференциальное уравнение в частных производных решается с помощью преобразования Лапласа. Нестационарное значение теплового потока получено как функция времени и координаты в канале при $n = 1$–6. Для сравнения также найдены решения для ряда приближенных моделей, а именно, чистой теплопроводности, ползучего и квазистационарного течений, и для новой модели, предложенной Сасеком.